Applications of Residue Calculus

Thursday, November 2, 2023 11:00 AM

1) Computing integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

Substitute: $d\theta = -i \dfrac{de^{i\theta}}{e^{i\theta}}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$-i \int_0^{2\pi} R\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right) \frac{de^{i\theta}}{e^{i\theta}} = \boxed{-i \oint_{C_1} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{z}}$$

$(C_1 = \{e^{i\theta}, 0 \le \theta \le 2\pi\}$.

compute it using residues if R is "nice" (rational!)

Example.

$$\int_{-\pi}^{\pi} \frac{\sin\theta}{1 - 2a\sin\theta + a^2}\, d\theta \qquad -1 < a < 1$$

$$-i \oint \frac{-\frac{i}{2}\left(z - \frac{1}{z}\right)}{1 + ai\left(z - \frac{1}{z}\right) + a^2} \frac{dz}{z} =$$

$$-\frac{1}{2} \oint \frac{(z^2-1)\, dz}{z^2 + aiz^3 - aiz + a^2z^2} = -\frac{1}{2}\oint \frac{(z^2-1)\,dz}{z(aiz^2 + (a^2+1)z - ai)} =$$

$$-\frac{1}{2}\cdot 2\pi i \left(\operatorname{Res}_{z=0} + \operatorname{Res}_{z=ai}\right) =$$

$$-\pi i \left(\frac{-1}{ai} + \frac{(z^2-1)}{z(aiz^2 + (a^2+1)z - ai)'}\Big|_{z=ai}\right) =$$

$$-\pi i \left(-\frac{i}{a} + \frac{-1-a^2}{ai(-2a^2 + a^2 + 1)}\right) = \frac{\pi a}{1-a^2}.$$

roots $z = 0, ai, -\frac{i}{a}$

$\operatorname{Res}_{z=0} \frac{f(z)}{z} = f(0)$

$f(z) = \dfrac{z^2-1}{aiz^2 + (a^2+1)z - ai}$

If g has simple zero at z_0, $f(z_0) \neq 0$

$$\operatorname{Res}_{z=z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

$f(z) = \dfrac{z^2-1}{z}$

$g(z) = aiz^2 + (a^2+1)z - ai$

$z_0 = ai$

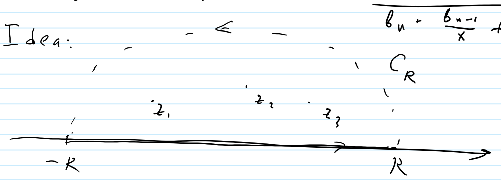
2) $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$, where P, Q - polynomials, $\deg Q \ge \deg P + 2$, $Q(x) \neq 0$, $x \in \mathbb{R}$.

Improper integral converges: for large x,

$$\frac{P(x)}{Q(x)} = \frac{\sum_{k=1}^{m} a_k x^k}{\sum_{k=1}^{n} b_k x^k} = x^{m-n} \frac{\sum a_k x^{k-m}}{\sum b_k x^{k-n}} \sim x^{m-n} \le x^{-2}.$$

$n = \deg Q$, $m = \deg P$ $\quad x^{m-n} \dfrac{a_m + \frac{a_{m-1}}{x} + \frac{a_{m-2}}{x^2} + \dots}{b_n + \frac{b_{n-1}}{x} + \dots}$

Idea:



$C_R = \{Re^{it}, 0 \le t \le \pi\}$ - upper half-circle.

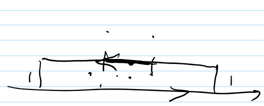
$\gamma_R := [-R, R] + C_R$. Choose R large: $R > \max\{|z| : Q(z) = 0\}$.

Then

$$\oint_{\gamma_R} \frac{P(z)}{Q(z)}\, dz = 2\pi i \left(\sum_{\substack{Q(z_j)=0 \\ \operatorname{Im} z_j > 0}} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}\right).$$

$$\| $$

$$\underbrace{\int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx}_{\text{I}} + \underbrace{\oint_{C_R} \frac{P(z)}{Q(z)}\, dz}_{\text{II}}$$



As $R \to \infty$, $\text{I} \to \displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$

As $R\to\infty$, $\quad I \to \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$

$$|II| \le \text{length}(C_R)\max_{|z|=R}\left|\frac{P(z)}{Q(z)}\right| \le 2\pi R\cdot C\frac{a_m/b_n}{R^2} \xrightarrow[R\to\infty]{} 0.$$

So, $$\boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}}$$

Example

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4} = 2\pi i\left(\sum_{\substack{z^4=-1\\ \operatorname{Im} z>0}} \operatorname{Res}_{z=z_j}\frac{z^2}{1+z^4}\right) = 2\pi i\left(\operatorname{Res}_{z=\frac{1+i}{\sqrt2}}\frac{z^2}{1+z^4} + \operatorname{Res}_{z=\frac{-1+i}{\sqrt2}}\frac{z^2}{1+z^4}\right) =$$

simple poles

$$2\pi i\left(\frac{z^2}{4z^3}\Big|_{z=\frac{1+i}{\sqrt2}} + \frac{z^2}{4z^3}\Big|_{z=\frac{i-1}{\sqrt2}}\right) = \frac{2\pi i}{4}\left(\frac{1-i}{\sqrt2} + \frac{-1-i}{\sqrt2}\right) = \frac{\pi}{2}\left(i\left(\frac{2(-i)}{\sqrt2}\right)\right) = \frac{\pi}{\sqrt2}$$

3) Integrals of the form

$$\int_{-\infty}^{\infty} \sin\lambda x\,\frac{P(x)}{Q(x)}\,dx \quad\text{and}\quad \int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx$$

$\lambda>0$

where P, Q - polynomials,

$\deg Q \ge \deg P+1$, $Q(x)\ne 0$ for $x\in\mathbb{R}$.

Note: does not always converge absolutely:
if $\deg Q = \deg P+1$, $\frac{P(x)}{Q(x)} \sim \frac{1}{x}$ for large x, and $\int^{\infty}\frac{dx}{x}$ diverges!

$\cos\lambda x = \operatorname{Re} e^{i\lambda x}$
$\sin\lambda x = \operatorname{Im} e^{i\lambda x}$, so, if P and Q have real coefficients, need to compute $\int_{-\infty}^{\infty} e^{i\lambda x}\frac{P(x)}{Q(x)}\,dx$.

Different approach than in Ahlfors: Jordan Lemma.

Camille Jordan

Theorem (Jordan Lemma).

Let f be a continuous function in the upper-half plane $\mathbb{H} := \{z : \operatorname{Im} z>0\}$. Let C_R be the semi-circle $\{|z|=R, \operatorname{Im} z>0\}$

$M_R := \sup_{z \in C_R} |f(z)|$. Assume $M(R) \to 0$ when $R \to \infty$ and $\lambda > 0$. Then

$$\int_{C_R} |f(z) e^{i\lambda z}| |dz| \underset{R\to\infty}{\longrightarrow} 0.$$

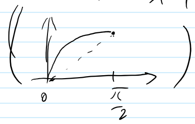
~~Direct "proof":~~ $\cancel{\int_{C_R} |f(z) e^{i\lambda z}| |dz| \le M(R) \max_{C_R} |e^{i\lambda z}| \, \pi R = \pi R M(R) \overset{?}{\to} 0}$

$\cancel{|e^{i\lambda(x+iy)}| = e^{-\lambda y} \le 1}$

Proof. $\int_{C_R} |f(z) e^{i\lambda z}| |dz| \le M_R \int_{C_R} |e^{i\lambda z}| |dz| = M_R \int_0^\pi |e^{i\lambda R(\cos\varphi + i\sin\varphi)}| R \, d\varphi =$

$$M_R R \int_0^\pi e^{-\lambda R \sin\varphi} d\varphi \underset{\sin(\pi-\varphi) = \sin\varphi}{=} 2 M_R R \int_0^{\pi/2} e^{-\lambda R \sin\varphi} d\varphi \le$$

$\varphi \ge \sin\varphi \ge \frac{2}{\pi}\varphi$ for $\varphi \in [0, \frac{\pi}{2}]$.



$$\le 2 M_R R \int_0^{\pi/2} e^{-\lambda R \frac{2}{\pi}\varphi} d\varphi = 2 M_R R \frac{\pi}{2R\lambda}\left(1 - e^{-\left(\frac{2\lambda R}{\pi}\cdot\frac{\pi}{2}\right)}\right) =$$

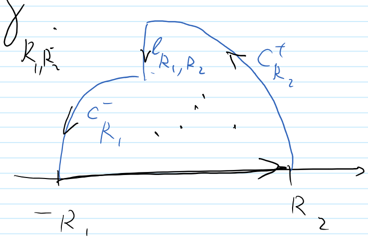
$$M_R \frac{\pi}{\lambda}(1 - e^{-\lambda R}) \le M_R \frac{\pi}{\lambda} \to 0 \quad \blacksquare$$

Application of Jordan Lemma:

$$\int_{-\infty}^{\infty} e^{i\lambda x} \frac{P(x)}{Q(x)} dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{i\lambda x} \frac{P(x)}{Q(x)} dx$$

$\cancel{\int_{-\infty}^{\infty} \frac{dx}{x} = 0}$ $\quad \cancel{\int_{-\infty}^{\infty} x \, dx = \lim_{R\to\infty} \int_{-R}^{R} x \, dx = 0}$

γ_{R_1, R_2}:



Take R_1, R_2, such that all zeroes z of Q, $\operatorname{Im} z > 0$, satisfy $|z| < \min(R_1, R_2)$.

Then $\oint_{\gamma_{R_1,R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_{\substack{Q(z_j) = 0 \\ \operatorname{Im} z_j > 0}} \operatorname{Res}_{z = z_j} e^{i\lambda z} \frac{P(z)}{Q(z)}$

But as $|z| \to \infty$ $\frac{|P(z)|}{|Q(z)|} \to 0$. So, by Jordan Lemma,

$$\left| \oint_{C^-_{R_1}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz \right| \le \int_{C_{R_1}} \left| e^{i\lambda z} \frac{|P(z)|}{|Q(z)|} \right| |dz| \to 0$$

$$\left| \oint_{C^+_{R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz \right| \to 0$$

and ($R_1 \le R_2$) $\left| \oint_{\ell_{R_1,R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz \right| = \left| \int_{R_1}^{R_2} e^{-\lambda y} \frac{P(iy)}{Q(iy)} dy \right| \le \frac{e^{-\lambda R_1}}{\lambda} \underbrace{\max_{y \ge R_1} \frac{P(iy)}{Q(iy)}}_{\to 0} \to 0$

So $\lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{i\lambda x} \frac{P(x)}{Q(x)} dx$ exists. So

So $\lim\limits_{R_1,R_2\to\infty}\int_{-R_1}^{R_2} e^{i\lambda x}\frac{P(x)}{Q(x)}\,dx$ exists. So

$$\int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx = \mathrm{Re}\Big(2\pi i\sum_{\substack{Q(z_j)=0\\ \mathrm{Im}\, z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)$$

$$\int_{-\infty}^{\infty}\sin\lambda x\,\frac{P(x)}{Q(x)}\,dx = \mathrm{Im}\Big(2\pi i\sum_{\substack{Q(z_j)=0\\ \mathrm{Im}\, z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)$$

both integrals converge!

<u>Example.</u>

$$\int_{-\infty}^{\infty}\frac{\sin\lambda x\cdot(x^3+1)}{(x^4+1)}\,dx = \mathrm{Im}\Big(2\pi i\Big(\operatorname{Res}_{z=\frac{1+i}{\sqrt2}}\Big(e^{i\lambda z}\frac{z^3+1}{z^4+1}\Big)+\operatorname{Res}_{z=\frac{-1+i}{\sqrt2}}\Big(e^{i\lambda z}\frac{z^3+1}{z^4+1}\Big)\Big)\Big)=$$

$$\mathrm{Im}\Big(2\pi i\Big(\Big(\frac{e^{i\lambda z}(z^3+1)}{4z^3}\Big)\Big|_{z=\frac{1+i}{\sqrt2}}+\Big(\frac{e^{i\lambda z}(z^3+1)}{4z^3}\Big)\Big|_{z=\frac{-1+i}{\sqrt2}}\Big)\Big)=$$

$$\mathrm{Im}\Big(\frac{\pi i}{2}\Big(e^{i\lambda\left(\frac{1+i}{\sqrt2}\right)}\Big(1-\frac{1+i}{\sqrt2}\Big)+e^{i\lambda\left(\frac{i-1}{\sqrt2}\right)}\Big(1-\frac{1-i}{\sqrt2}\Big)\Big)\Big)=$$

$\frac{1}{z^3}=-z$ $\left(z=\frac{i+1}{\sqrt2}\right)$

$$\frac{\pi}{2}e^{-\lambda/\sqrt2}\Big(\cos\frac{\lambda}{\sqrt2}\Big(1-\frac{1}{\sqrt2}\Big)-\frac{\sin\frac{\lambda}{\sqrt2}}{\sqrt2}+\cos\frac{\lambda}{\sqrt2}\Big(1-\frac{1}{\sqrt2}\Big)-\frac{\sin\frac{\lambda}{\sqrt2}}{\sqrt2}\Big)=$$

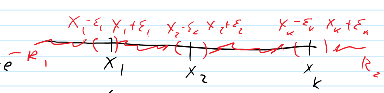
$$\boxed{\frac{\pi}{2}e^{-\lambda/\sqrt2}\Big(\cos\frac{\lambda}{\sqrt2}\Big)(2-\sqrt2)}$$

4) $\displaystyle\int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx$ $\displaystyle\int_{-\infty}^{\infty}\sin\lambda x\,\frac{P(x)}{Q(x)}\,dx$, where $\deg Q>\deg P$, Q – has simple zeroes on $\mathbb{R}$.

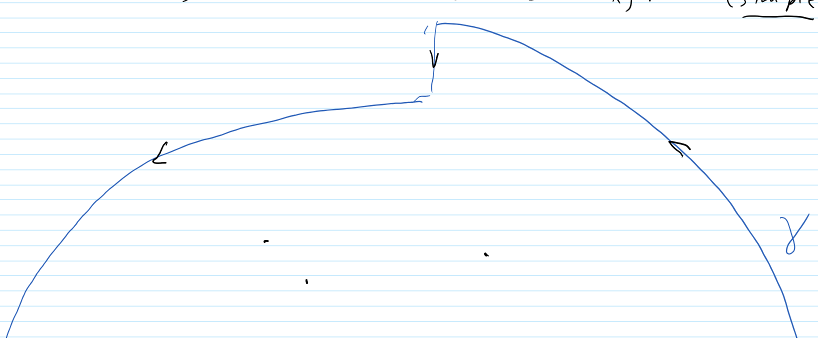
The integrals diverge near zeroes!
But we can consider <u>principle value</u>.

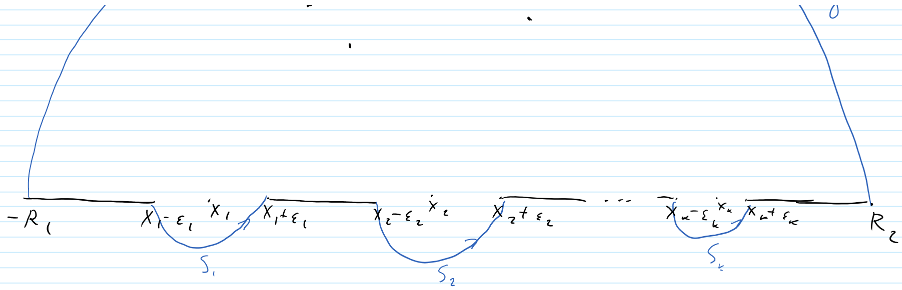
$$\text{p.v.}\int_{-\infty}^{\infty}\begin{matrix}\cos\lambda x\\ \sin\lambda x\end{matrix}\,\frac{P(x)}{Q(x)}\,dx=\lim_{\substack{R_1,R_2\to\infty\\ \varepsilon_1,\varepsilon_2,\dots\varepsilon_k\to 0}}\int_{-R_1}^{x_1-\varepsilon_1}+\int_{x_1+\varepsilon_1}^{x_2-\varepsilon_2}+\dots\int_{x_k+\varepsilon_k}^{R_2}\dots$$

; where $x_1<\dots<x_k$ – zeroes of Q. (simple)



We remove symmetric intervals around x_j!





Observe: $\oint_\gamma e^{i\lambda z}\frac{P(z)}{Q(z)}\,dz = 2\pi i\Bigg(\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}} \operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}e^{i\lambda z} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}} \operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}e^{i\lambda z}\Bigg).$

By Jordan Lemma, $\int_{C_R} \to 0.$

What about $\oint_{S_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\,dz$?

x_j is a simple pole, so $e^{i\lambda z}\frac{P(z)}{Q(z)} = \frac{c_{-1}}{z-x_j}+g(z)$, where g is holomorphic at x_j. $c_{-1} = \operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}.$

Then $\oint_{S_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\,dz = \underbrace{\oint_{S_j}\frac{c_{-1}}{z-x_j}\,dz}_{I} + \underbrace{\oint_{S_j} g(z)\,dz}_{II}$

$II \le \max|g(z)|\,\ell(S_j) \xrightarrow[\varepsilon_j\to 0]{} 0$

$$I = \int_0^\pi \frac{c_{-1}}{(x_j-\varepsilon_j e^{it})-x_j}\,d(x_j-\varepsilon_j e^{it}) = \pi i\, c_{-1} = \pi i \operatorname{Res}_{z=x_j} e^{i\lambda z}\frac{P(z)}{Q(z)}.$$

So $\int_{S_j} \to \pi i \operatorname{Res}_{z=x_j} e^{i\lambda z}\frac{P(z)}{Q(z)}$

Plugging in the formula, we get

$$\text{p.v.}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{P(x)}{Q(x)}\,dx = \pi i\Bigg(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Bigg)$$

$$\text{p.v.}\int_{-\infty}^{\infty} \cos\lambda x\,\frac{P(x)}{Q(x)}\,dx = \operatorname{Re}\Bigg(\pi i\Bigg(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Bigg)\Bigg)$$

$$\text{p.v.}\int_{-\infty}^{\infty} \sin\lambda x\,\frac{P(x)}{Q(x)}\,dx = \operatorname{Im}\Bigg(\pi i\Bigg(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Bigg)\Bigg)$$

Example:

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\sin\lambda x}{1+x^3}\,dx = \operatorname{Im}\left(\pi i\left(2\operatorname{Res}_{z=e^{\frac{\pi i}{3}}}\frac{e^{i\lambda z}}{1+z^3} + \operatorname{Res}_{z=-1}\frac{e^{i\lambda z}}{1+z^3}\right)\right) =$$

$$\operatorname{Im}\left(\pi i\left(2\,\frac{e^{i\lambda(\frac12+\frac{\sqrt3}{2}i)}}{3(-\frac12+\frac{\sqrt3}{2}i)} + \frac{e^{-\lambda}}{3}\right)\right) =$$

$$\operatorname{Im}\left(\pi i\left(\frac23\left(e^{-\frac{\sqrt3}{2}\lambda}\left(\cos\frac{\lambda}{2}+i\sin\frac{\lambda}{2}\right)\right)\left(-\frac12-\frac{\sqrt3}{2}i\right)+\frac{e^{-\lambda}}{3}\right)\right) =$$

$$-\frac13\pi\left(e^{-\frac{\sqrt3}{2}\lambda}\cos\frac{\lambda}{2} + \frac{\sqrt3}{3}e^{-\frac{\sqrt3}{2}\lambda}\sin\frac{\lambda}{2} + \frac{e^{-\lambda}}{3}\right).$$